OCR

RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MATHEMATICS **4727**

Further Pure Mathematics 3

Wednesday **25 JANUARY 2006** Morning 1 hour 30 minutes

Additional materials:
8 page answer booklet
Graph paper
List of Formulae (MF1)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.
- **You are reminded of the need for clear presentation in your answers.**

This question paper consists of 3 printed pages and 1 blank page.

1 Find the acute angle between the skew lines

$$\frac{x+3}{1}=\frac{y-2}{1}=\frac{z-4}{-1} \quad \text{and} \quad \frac{x-5}{2}=\frac{y-1}{-3}=\frac{z+3}{1}.$$ [4]

2 The tables shown below are the operation tables for two isomorphic groups G and H.

G	a	b	c	d
a	d	a	b	c
b	a	b	c	d
c	b	c	d	a
d	c	d	a	b

H	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

(i) For each group, state the identity element and list the elements of any proper subgroups. [4]

(ii) Establish the isomorphism between G and H by showing which elements correspond. [3]

3 **(i)** By using the substitution $y^3 = z$, find the general solution of the differential equation

$$3y^2\frac{dy}{dx} + 2xy^3 = e^{-x^2},$$

giving y in terms of x in your answer. [6]

(ii) Describe the behaviour of y as $x \to \infty$. [1]

4 **(i)** By expressing $\cos\theta$ and $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, or otherwise, show that

$$\cos^2\theta\sin^4\theta = \tfrac{1}{32}(\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2).$$ [6]

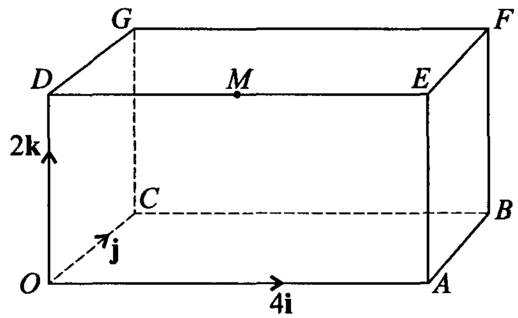
(ii) Hence find the exact value of

$$\int_0^{\frac{1}{3}\pi} \cos^2\theta\sin^4\theta\, d\theta.$$ [3]

5 **(i)** Solve the equation $z^4 = 64(\cos\pi + i\sin\pi)$, giving your answers in polar form. [2]

(ii) By writing your answers to part **(i)** in the form $x + iy$, find the four linear factors of $z^4 + 64$. [4]

(iii) Hence, or otherwise, express $z^4 + 64$ as the product of two real quadratic factors. [3]

6 

The cuboid $OABCDEFG$ shown in the diagram has $\overrightarrow{OA} = 4\mathbf{i}$, $\overrightarrow{OC} = \mathbf{j}$, $\overrightarrow{OD} = 2\mathbf{k}$, and M is the mid-point of DE.

(i) Find a vector perpendicular to $\overrightarrow{MB}$ and $\overrightarrow{OF}$. [3]

(ii) Find the cartesian equations of the planes CMG and OEG. [5]

(iii) Find an equation of the line of intersection of the planes CMG and OEG, giving your answer in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. [3]

7 A group G has an element a with order n, so that $a^n = e$, where e is the identity. It is given that x is any element of G distinct from a and e.

(i) Prove that the order of $x^{-1}ax$ is n, making it clear which group property is used at each stage of your proof. [6]

(ii) Express the inverse of $x^{-1}ax$ in terms of some or all of x, x^{-1}, a and a^{-1}, showing sufficient working to justify your answer. [3]

(iii) It is now given that a commutes with every element of G. Prove that a^{-1} also commutes with every element. [2]

8 **(i)** Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 4x = 0,$$

where k is a real constant, in each of the following cases.

(a) $|k| > 2$

(b) $|k| < 2$

(c) $k = 2$

[8]

(ii) (a) In the case when $k = 1$, find the solution for which $x = 0$ and $\frac{dx}{dt} = 6$ when $t = 0$. [4]

(b) Describe what happens to x as $t \to \infty$ in this case, justifying your answer. [2]

1 Directions $[1, 1, -1]$ and $[2, -3, 1]$	B1	For identifying both directions (may be implied by working)
$\theta = \cos^{-1}\dfrac{\lvert[1,1,-1].[2,-3,1]\rvert}{\sqrt{3}\ \ \sqrt{14}}$	M1	For using scalar product of their direction vectors
$= \cos^{-1}\dfrac{\lvert -2\rvert}{\sqrt{42}}$	M1	For completely correct process for their angle
$= 72.0°, 72°$ or 1.26 rad	A1 4 [4]	For correct answer
2 (i) Identities b, 6 Subgroups $\{b, d\}$, $\{6, 4\}$	B1 B1 B1 B1 4	For correct identities For correct subgroups
(ii) $\{a, b, c, d\} \leftrightarrow \{2, 6, 8, 4\}$ or $\{8, 6, 2, 4\}$	B1 B1 B1 3 [7]	For $b \leftrightarrow 6$, $d \leftrightarrow 4$ For $a, c \leftrightarrow 2, 8$ in either order **SR** If B0 B0 B0 then M1 A1 may be awarded for stating the orders of all elements in G and H
3 (i) $3y^2\dfrac{dy}{dx} = \dfrac{dz}{dx}$	M1	For differentiating substitution
$\Rightarrow \dfrac{dz}{dx} + 2xz = e^{-x^2}$	A1	For resulting equation in z and x
Integrating factor $\left(e^{\int 2x\,dx} =\right) e^{x^2}$	B1 √	For correct IF f.t. for an equation in suitable form
$\Rightarrow \dfrac{d}{dx}\left(ze^{x^2}\right)$ *OR* $\dfrac{d}{dx}\left(y^3e^{x^2}\right) = 1$	M1	For using IF correctly
$\Rightarrow ze^{x^2}$ *OR* $y^3e^{x^2} = x\,(+c)$	A1	For correct integration ($+c$ not required here)
$\Rightarrow y = (x+c)^{\frac{1}{3}}\,e^{-\frac{1}{3}x^2}$	A1 6	For correct answer **AEF**
(ii) As $x \to \infty$, $y \to 0$	B1 1 [7]	For correct statement
4 (i) $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$, $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$	B1	For either expression, seen or implied z may be used for $e^{i\theta}$ throughout
$\Rightarrow \cos^2\theta\sin^4\theta = \frac{1}{4}\left(e^{i\theta} + e^{-i\theta}\right)^2 \frac{1}{16}\left(e^{i\theta} - e^{-i\theta}\right)^4$		
$= \frac{1}{4}\left(e^{2i\theta} + 2 + e^{-2i\theta}\right).\frac{1}{16}\left(e^{4i\theta} - 4e^{2i\theta} + 6 - 4e^{-2i\theta} + e^{-4i\theta}\right)$	M1 A1 A1	For expanding terms For the 2 correct expansions **SR** Allow A1 A0 for $k\left(e^{2i\theta} + 2 + e^{-2i\theta}\right)\left(e^{4i\theta} - 4e^{2i\theta} + 6 - 4e^{-2i\theta} + e^{-4i\theta}\right)$, $k \neq \frac{1}{64}$
$= \frac{1}{64}\left(\left(e^{6i\theta} + e^{-6i\theta}\right) - 2\left(e^{4i\theta} + e^{-4i\theta}\right) - \left(e^{2i\theta} + e^{-2i\theta}\right) + 4\right)$	M1	For grouping terms and using multiple angles
$= \frac{1}{32}(\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2)$ **AG**	A1 6	For answer obtained correctly

(ii) $\int_0^{\frac{1}{3}\pi}\cos^2\theta\sin^4\theta\,d\theta =$		
$=\frac{1}{32}\left[\frac{1}{6}\sin 6\theta-\frac{1}{2}\sin 4\theta-\frac{1}{2}\sin 2\theta+2\theta\right]_0^{\frac{1}{3}\pi}$	M1 A1	For integrating answer to **(i)** For all terms correct
$=\frac{1}{32}\left[0+\frac{1}{4}\sqrt{3}-\frac{1}{4}\sqrt{3}+\frac{2}{3}\pi-0\right]=\frac{1}{48}\pi$	A1 **3** 9	For correct answer
5 (i) *EITHER* $z=\sqrt{8}\,\text{cis}(2k+1)\frac{\pi}{4},\ k=0,1,2,3$	B1	For correct modulus **AEF**
OR $z=\sqrt{8}\,e^{(2k+1)\frac{\pi}{4}i},\ k=0,1,2,3$	B1 **2**	For correct arguments **AEF**
(ii) $z=2\sqrt{2}\left\{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i,-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i,-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i,\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right\}$	B1	For any of $\pm\frac{1}{\sqrt{2}}\pm\frac{1}{\sqrt{2}}i$
$z=2+2i,\ -2+2i,\ -2-2i,\ 2-2i$	B1	For any one value of z correct
	B1	For all values of z correct **AEFcartesian** (may be implied from symmetry or factors)
$(z-\alpha),(z-\beta),(z-\gamma),(z-\delta)$	B1 √ **4**	f.t., where $\alpha,\beta,\gamma,\delta$ are answers above
(iii) *EITHER* $(z-(2+2i))(z-(2-2i))$	M1	For combining factors from **(ii)** in pairs
$\times(z-(-2+2i))(z-(-2-2i))$	M1	Use of complex conjugate pairs
$=(z^2+4z+8)(z^2-4z+8)$	A1	For correct answer
OR $z^4+64=(z^2+az+b)(z^2+cz+d)$	M1	For equating coefficients
$\Rightarrow a+c=0,\ b+ac+d=0,\ ad+bc=0,\ bd=64$	M1	For solving equations
Obtain $(z^2+4z+8)(z^2-4z+8)$	A1 **3** 9	For correct answer
6 (i) **MB** $=[2,1,-2]$, **OF** $=[4,1,2]$	B1	For either vector correct (allow multiples)
MB × **OF**	M1	For finding vector product of their **MB** and **OF**
$=[4,-12,-2]$ *OR* $k[2,-6,-1]$	A1 **3**	For correct vector
(ii) *EITHER* Find vector product of any two of $\pm[2,-1,2]$, $\pm[0,0,2]$, $\pm[2,-1,0]$ and any two of $\pm[4,0,2]$, $\pm[4,-1,0]$, $\pm[0,1,2]$	M1	For finding two relevant vector products
Obtain $k[1,2,0]$	A1	For correct LHS of plane *CMG*
Obtain $k[1,4,-2]$	A1	For correct LHS of plane *OEG*
	M1	For substituting a point into each equation
$x+2y=2$ and $x+4y-2z=0$	A1	For both equations correct **AEF**
OR Use $ax+by+cz=d$ with coordinates of *C*, *M*, *G* *OR* *O*, *E*, *G* substituted	M1	For use of cartesian equation of plane
Obtain $a:b:c=1:2:0$ for *CMG*	A1	For correct ratio
Obtain $a:b:c=1:4:-2$ for *OEG*	A1	For correct ratio
	M1	For substituting a point into each equation
$x+2y=2$ and $x+4y-2z=0$	A1 **5**	For both equations correct **AEF**

(iii) *EITHER* Put x, y *OR* $z = t$ in planes *OR* evaluate $k[1, 2, 0] \times k[1, 4, -2]$ Obtain $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ where $\mathbf{a} = [0, 1, 2], [2, 0, 1]$ *OR* $[4, -1, 0]$ $\mathbf{b} = k[-2, 1, 1]$	M1 A1 A1 **3** **11**	For solving plane equations in terms of a parameter *OR* for finding vector product of normals to planes from **(ii)** Obtain a correct point **AEF** Obtain correct direction **AEF**
7 (i) $(x^{-1}ax)^m = (x^{-1}ax)(x^{-1}ax)\ldots(x^{-1}ax)$ $= x^{-1}aa\ldots ax$, associativity, $xx^{-1} = e$ $= x^{-1}a^m x = x^{-1}ex$ when $m = n$, not $m < n$ $= x^{-1}x$ $= e \Rightarrow$ order n	M1 A1 A1 B1 A1 A1 **6**	For considering powers of $x^{-1}ax$ For using associativity and inverse properties For using order of *a* correctly For using property of identity For correct conclusion
(ii) ***EITHER*** $(x^{-1}ax)z = e$ $\Rightarrow axz = xe = x \Rightarrow xz = a^{-1}x$ $\Rightarrow z = x^{-1}a^{-1}x$	M1 A1 A1	For attempt to solve for *z* **AEF** For using pre- or post multiplication For correct answer
OR Use $(pq)^{-1} = q^{-1}p^{-1}$ *OR* $(pqr)^{-1} = r^{-1}q^{-1}p^{-1}$ State $(x^{-1})^{-1} = x$ Obtain $x^{-1}a^{-1}x$	M1 A1 A1 **3**	For applying inverse of a product of elements For stating this property For correct answer with no incorrect working **SR** correct answer with no working scores B1 only
(iii) $ax = xa \Rightarrow x = a^{-1}xa$ $\Rightarrow xa^{-1} = a^{-1}x$	M1 A1 **2** **11**	Start from commutative property for *ax* Obtain commutative property for $a^{-1}x$
8 (i) $m^2 + 2km + 4 = 0$ $\Rightarrow m = -k \pm \sqrt{k^2 - 4}$ **(a)** $x = e^{-kt}\left(Ae^{\sqrt{k^2-4}\,t} + Be^{-\sqrt{k^2-4}\,t}\right)$	M1 A1 **2** M1 A1 **2**	For stating and attempting to solve auxiliary eqn For correct solutions, at any stage **AEF** For using $e^{f(t)}$ with distinct real roots of aux eqn For correct answer **AEF**
(b) $x = e^{-kt}\left(Ae^{i\sqrt{4-k^2}\,t} + Be^{-i\sqrt{4-k^2}\,t}\right)$ $x = e^{-kt}\left(A'\cos\sqrt{4-k^2}\,t + B'\sin\sqrt{4-k^2}\,t\right)$ *OR* $x = e^{-kt}\left(C' \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\left(\sqrt{4-k^2}\,t + \alpha\right)\right)$	M1 A1 **2**	For using $e^{f(t)}$ with complex roots of aux eqn This form may not be seen explicitly but if stated as final answer earns M1 A0 For correct answer
(c) $x = e^{-2t}(A'' + B''t)$	M1 A1 **2**	For using $e^{f(t)}$ with equal roots of aux eqn For correct answer. Allow *k* for 2

(ii)(a) $x = B'e^{-t}\sin\sqrt{3}\,t$	B1 √	For using $t = 0, x = 0$ correctly. f.t. from **(b)**
$\dot{x} = B'e^{-t}\left(\sqrt{3}\cos\sqrt{3}\,t - \sin\sqrt{3}\,t\right)$	M1 A1 √	For differentiating x For correct expression. f.t. from their x
$t = 0, \dot{x} = 6 \Rightarrow B' = 2\sqrt{3}$, $x = 2\sqrt{3}e^{-t}\sin\sqrt{3}\,t$	A1 **4**	For correct solution **AEF** **SR** √ and **AEF** OK for $x = C'e^{-t}\cos\left(\sqrt{3}\,t + \frac{1}{2}\pi\right)$
(b) $x \to 0$	B1	For correct statement
$e^{-t} \to 0$ and sin( ) is bounded	B1 **2** 14	For both statements